

NOTE ON THE BONDAL-ORLOV FUNCTORS FOR TORIC DM STACKS

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ABSTRACT. We calculate explicit formulas for the general equivariant Bondal-Orlov functors on the localized K-theory groups for a crepant birational transformation of toric DM stacks. We recall some facts that the Bondal-Orlov functors give equivalences on the bounded derived categories. Applying twice of these functors we get the Seidel-Thomas spherical twists for the derived category.

1. INTRODUCTION

In this short note we calculate explicit formulas for the general equivariant Bondal-Orlov functors for a crepant birational transformation of toric Deligne–Mumford (DM) stacks.

Toric DM stacks were introduced by Borisov–Chen–Smith [2] using stacky fans. The notion of extended stacky fan was introduced by Jiang in [12], and it turns out that there is a one-to-one correspondence between the extended stacky fans and GIT data construction of toric DM stacks. Given GIT data determined by a stability parameter ω , we denote the toric DM stack by X_ω , whose construction is reviewed in § 2.1. More details can be found in [8]. Birational transformation of toric DM stacks can be understood as changing the GIT stability parameters in the space of GIT stability conditions.

We study a special case of birational transformation of toric DM stacks: the *crepant* birational transformations. We consider a special class of crepant birational transformations (K -equivalences) of toric DM stacks by a single wall crossing. The construction of such wall crossing can be found in [8, § 5.1]. There is a big torus T action on the toric DM stack X_ω , and we work on the T -equivariant K -theory and bounded derived category on X_ω . Y. Kawamata in [14] proves that a natural Fourier–Mukai transform induces equivalences of the bounded derived categories of K -equivalent toric DM stacks. It was shown in [9] that the T -equivariant derived categories are also equivalent. In [8, § 6], the authors calculated the equivariant Fourier–Mukai transform for K -theory basis of X_ω when restricted to torus fixed points.

In this paper we calculate explicit formulas for the general Bondal-Orlov functors in terms of equivariant K -theory basis for a single toric wall crossing. Let

$$\varphi : X_+ := X_{\omega_+} \dashrightarrow X_- := X_{\omega_-}$$

be a crepant transformation by a single wall crossing corresponding to the stability conditions ω_+ and ω_- . The T -equivariant K -theory $K_0^T(X_\pm)$ are generated by equivariant line bundles corresponding to the lattice in the secondary fan. There is a common blow-up \tilde{X} for both X_+ and X_- and two contract maps $f_\pm : \tilde{X} \rightarrow X_\pm$.

Let $E \subset \tilde{X}$ be the exceptional divisor. The general Bondal-Orlov functors are defined by:

$$(1.1) \quad \mathbb{B}\mathcal{O}_k = (f_+)_*(\mathcal{O}_{\tilde{X}}(kE) \otimes (f_-)^*(-)) : D_T^b(X_-) \rightarrow D_T^b(X_+)$$

for any integer $k \in \mathbb{Z}$. We prove that $\mathbb{B}\mathcal{O}_k$ is an equivalence on the equivariant bounded derived categories for any k . When $k = 0$, $\mathbb{B}\mathcal{O}_0$ is the usual Fourier-Mukai transform $\mathbb{F}\mathcal{M} = (f_+)_*((f_-)^*(-))$. So $\mathbb{B}\mathcal{O}_k$ can be taken as *generalized* Fourier-Mukai transforms. The functors $\mathbb{B}\mathcal{O}_k$, of course, induce isomorphisms on the equivariant K -theory groups. Our computation gives explicit formulas of the Bondal-Orlov functors $\mathbb{B}\mathcal{O}_k$ on the localized K -theory basis. See Theorem 3.2. This generalizes the calculation of Theorem 6.19 in [8] for the Fourier-Mukai transform $\mathbb{B}\mathcal{O}_0 = \mathbb{F}\mathcal{M}$, although the proof is basically the same as in [8]. In Theorem 6.23 of [8], the authors prove that the Fourier-Mukai transform $\mathbb{B}\mathcal{O}_0$ matches the analytic continuation of the H -functions for X_{\pm} , which implies the invariance of big quantum cohomology of X_{\pm} , see [8, §5, 6] for details. It is pretty interesting if the general Bondal-Orlov functors $\mathbb{B}\mathcal{O}_k$ can match the analytic continuation of some hypergeometric functions for X_{\pm} .

We also recall the fact that the Bondal-Orlov functors give an equivalence on the bounded derived categories of a single toric wall crossing. The proof is based on the method of window shifted functor for the derived categories under GIT quotients by [10], [1] and [18]. We completely follow the proof of §5 in [9]. Applying back for the Bondal-Orlov functor we get an autoequivalence of the bounded derived category which is called the spherical twist functor associated with a line bundle on the contraction locus in the sense of Seidel-Thomas in [19]. We also give a proof that for a crepant birational transformation of toric DM stacks via a single wall crossing, the contraction locus are always weighted projective stacks. This result is hidden somewhere in [8], but there is no explicit explanation. The result presented here is related to the monodromy conjecture in [6] for Gromov-Witten theory of symplectic smooth DM stacks, see [13]. The result of the spherical twists can also be applied to find a correspondence for the Chen-Ruan cohomology for quasi-simple orbifold flops, see [7].

This short note is organized as follows. In §2 we review the construction of the crepant transformation of toric DM stacks by a single wall crossing. We calculate the general equivariant Bondal-Orlov functor on the localized K -theory basis for the wall crossing of toric DM stacks in §3. In §4 we recall the fact that the general equivariant Bondal-Orlov functors give an equivalence on the bounded derived categories for the wall crossing of toric DM stacks, and relate them to spherical twist associated with line bundles on the contraction locus.

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2. CREPANT TRANSFORMATION OF TORIC DM STACKS

In this section we review some basic facts and establish notations. The main reference is [8].

2.1. Toric Deligne–Mumford stack and GIT quotient. An S -extended stacky fan is a quadruple $\Sigma = (\mathbf{N}, \Sigma, \beta, S)$, where:

- \mathbf{N} is a finitely generated abelian group (torsions allowed);
- Σ is a rational simplicial fan in $\mathbf{N} \otimes \mathbb{R}$;
- $\beta: \mathbb{Z}^m \rightarrow \mathbf{N}$ is a homomorphism; we write $b_i = \beta(e_i) \in \mathbf{N}$ for the image of the i th standard basis vector $e_i \in \mathbb{Z}^m$, and write \bar{b}_i for the image of b_i in $\mathbf{N} \otimes \mathbb{R}$;
- $S \subset \{1, \dots, m\}$ is a subset,

such that:

- each one-dimensional cone of Σ is spanned by \bar{b}_i for a unique $i \in \{1, \dots, m\} \setminus S$, and each \bar{b}_i with $i \in \{1, \dots, m\} \setminus S$ spans a one-dimensional cone of Σ ;
- for $i \in S$, \bar{b}_i lies in the support $|\Sigma|$ of the fan.

The vectors b_i for $i \in S$ are called *extended vectors*.

The *toric DM stack* associated to an extended stacky fan $(\mathbf{N}, \Sigma, \beta, S)$ depends only on the underlying stacky fan and is defined as the quotient stack

$$X_\Sigma := [U/K], \quad \text{with } U = \mathbb{C}^m \setminus \mathbb{V}(I_\Sigma),$$

where I_Σ is the irrelevant ideal of the fan and $K := \mathbb{H}om(\mathbb{L}^\vee, \mathbb{C}^\times)$ acts on \mathbb{C}^m through the data of extended stacky fan.

We require that the extended stacky fans $(\mathbf{N}, \Sigma, \beta, S)$ satisfy the following conditions:

- (C1) the support $|\Sigma|$ of the fan is convex and full-dimensional;
- (C2) there is a strictly convex piecewise-linear function $f: |\Sigma| \rightarrow \mathbb{R}$ that is linear on each cone of Σ ;
- (C3) the map $\beta: \mathbb{Z}^m \rightarrow \mathbf{N}$ is surjective.

The first two conditions are geometric constraints on X_Σ : they are equivalent to saying that the corresponding toric stack X_Σ is semi-projective and has a torus fixed point. The third condition can be always achieved by adding enough extended vectors.

We explain the GIT construction of X_Σ from the extended stacky fan $\Sigma = (\mathbf{N}, \Sigma, \beta, S)$ satisfying (C1-C3). First we define a free \mathbb{Z} -module \mathbb{L} by the exact sequence

$$(2.1) \quad 0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^m \xrightarrow{\beta} \mathbf{N} \longrightarrow 0$$

and define $K := \mathbb{L} \otimes \mathbb{C}^\times$. The dual of (2.1) is an exact sequence:

$$(2.2) \quad 0 \longrightarrow \mathbf{N}^\vee \longrightarrow (\mathbb{Z}^m)^\vee \longrightarrow \mathbb{L}^\vee$$

and we define the character $D_i \in \mathbb{L}^\vee$ of K to be the image of the i th standard basis vector in $(\mathbb{Z}^m)^\vee$ under the third arrow $(\mathbb{Z}^m)^\vee \rightarrow \mathbb{L}^\vee$. Set

$$\mathcal{A}_\omega = \{I \subset \{1, 2, \dots, m\} \mid S \subset I, \sigma_I \text{ is a cone of } \Sigma\}.$$

to be the collection of *anticones*. The *stability condition* $\omega \in \mathbb{L}^\vee \otimes \mathbb{R}$ lies in $\bigcap_{I \in \mathcal{A}_\omega} \angle_I$, where

$$\angle_I = \{ \sum_{i \in I} a_i D_i \mid a_i \in \mathbb{R}, a_i > 0 \} \subset \mathbb{L}^\vee \otimes \mathbb{R}.$$

The condition (C2) ensures that this intersection is non-empty. We understand $\angle_\emptyset = \{0\}$. Let

$$U_\omega = \bigcup_{I \in \mathcal{A}_\omega} (\mathbb{C}^\times)^I \times \mathbb{C}^{\bar{I}} := (\mathbb{C}^\times)^I \times \mathbb{C}^{\bar{I}} = \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid z_i \neq 0 \text{ for } i \in I\}.$$

The GIT data consists of

- $K \cong (\mathbb{C}^\times)^r$, a connected torus of rank r ;
- $\mathbb{L} = \mathbb{H}om(\mathbb{C}^\times, K)$, the cocharacter lattice of K ;
- $D_1, \dots, D_m \in \mathbb{L}^\vee = \mathbb{H}om(K, \mathbb{C}^\times)$, characters of K ;
- stability condition $\omega \in \mathbb{L}^\vee \otimes \mathbb{R}$;
- $\mathcal{A}_\omega = \{I \subset \{1, 2, \dots, m\} : \omega \in \angle_I\}$.

The stability condition ω satisfies the following assumptions:

- Assumption 2.1.* (A1) $\{1, 2, \dots, m\} \in \mathcal{A}_\omega$;
 (A2) for each $I \in \mathcal{A}_\omega$, the set $\{D_i : i \in I\}$ spans $\mathbb{L}^\vee \otimes \mathbb{R}$ over \mathbb{R} .

(A1) ensures that X_ω is non-empty; (A2) ensures that X_ω is a DM stack. Under these assumptions, \mathcal{A}_ω is closed under enlargement of sets; i.e., if $I \in \mathcal{A}_\omega$ and $I \subset J$ then $J \in \mathcal{A}_\omega$. The toric DM stack is the quotient stack $X_\Sigma = X_\omega = [U_\omega/K]$.

Conversely, to obtain an extended stacky fan from GIT data, consider the exact sequence (2.1). Let $b_i = \beta(e_i) \in \mathbb{N}$ and $\bar{b}_i \in \mathbb{N} \otimes \mathbb{R}$ be as above and, given a subset I of $\{1, \dots, m\}$, let σ_I denote the cone in $\mathbb{N} \otimes \mathbb{R}$ generated by $\{\bar{b}_i : i \in I\}$. The extended stacky fan $\Sigma_\omega = (\mathbb{N}, \Sigma_\omega, \beta, S)$ corresponding to our data consists of the group \mathbb{N} and the map β defined above, together with a fan Σ_ω in $\mathbb{N} \otimes \mathbb{R}$ and S given by

$$\Sigma_\omega = \{\sigma_I : \bar{I} \in \mathcal{A}_\omega\}, \quad S = \{i \in \{1, \dots, m\} : \bar{b}_i \notin \mathcal{A}_\omega\}.$$

The quotient construction in [12, §2] coincides with the GIT quotient construction, and therefore X_ω is the toric DM stack corresponding to Σ_ω .

2.2. Wall crossing and birational transformation. The space $\mathbb{L}^\vee \otimes \mathbb{R}$ of stability conditions is divided into chambers by the closures of the sets \angle_I , $|I| = r - 1$, and the DM stack X_ω depends on ω only via the chamber containing ω . For any stability condition ω , the set U_ω contains the big torus $T = (\mathbb{C}^\times)^m$. Thus for any two such stability conditions ω_1, ω_2 there is a canonical birational map $X_{\omega_1} \dashrightarrow X_{\omega_2}$, induced by the identity transformation between $T/K \subset X_{\omega_1}$ and $T/K \subset X_{\omega_2}$.

Let C_+, C_- be chambers in $\mathbb{L}^\vee \otimes \mathbb{R}$ that are separated by a hyperplane wall W , so that $W \cap \bar{C}_+$ is a facet of \bar{C}_+ , $W \cap \bar{C}_-$ a facet of \bar{C}_- , and $W \cap \bar{C}_+ = W \cap \bar{C}_-$. Choose stability conditions $\omega_+ \in C_+, \omega_- \in C_-$ satisfying (A1-A2) and set $U_+ := U_{\omega_+}, U_- := U_{\omega_-}, X_+ := X_{\omega_+}, X_- := X_{\omega_-}$, and

$$\mathcal{A}_\pm := \mathcal{A}_{\omega_\pm} = \{I \subset \{1, 2, \dots, m\} : \omega_\pm \in \angle_I\}.$$

Then $C_\pm = \bigcap_{I \in \mathcal{A}_\pm} \angle_I$. Let $\varphi: X_+ \dashrightarrow X_-$ be the birational transformation induced by the toric wall-crossing from C_+ to C_- and suppose that $\sum_{i=1}^m D_i \in W$ which implies that φ is crepant. Let $e \in \mathbb{L}$ denote the *primitive lattice vector* in W^\perp such that e is positive on C_+ and negative on C_- . We fix the notations

- $M_+ := \{i \in \{1, \dots, m\} \mid D_i \cdot e > 0\}$,
- $M_- := \{i \in \{1, \dots, m\} \mid D_i \cdot e < 0\}$,
- $M_0 := \{i \in \{1, \dots, m\} \mid D_i \cdot e = 0\}$.

Choose ω_0 from the relative interior of $W \cap \overline{C_+} = W \cap \overline{C_-}$. The stability condition ω_0 does not satisfy (A1-A2) on GIT data, but consider

$$\mathcal{A}_0 := \mathcal{A}_{\omega_0} = \{I \subset \{1, \dots, m\} : \omega_0 \in \angle_I\}$$

and the corresponding toric Artin stack $X_0 := X_{\omega_0} = [U_{\omega_0}/K]$. Here X_0 is not a DM stack, as the \mathbb{C}^\times -subgroup of K corresponding to $e \in \mathbb{L}$ (the defining equation of the wall W) has a fixed point in $U_0 := U_{\omega_0}$. The stack X_0 contains both X_+ and X_- as open substacks and the canonical line bundles of X_+ and X_- are the restrictions of the same line bundle $L_0 \rightarrow X_0$ given by the character $-\sum_{i=1}^m D_i$ of K . The condition $\sum_{i=1}^m D_i \in W$ ensures that L_0 comes from a \mathbb{Q} -Cartier divisor on the underlying singular toric variety $\overline{X}_0 = \mathbb{C}^m //_{\omega_0} K$. There are canonical blow-down maps $g_\pm: X_\pm \rightarrow \overline{X}_0$, and $K_{X_\pm} = g_\pm^* L_0$. We have a commutative diagram:

(2.3)

$$\begin{array}{ccc} & \tilde{X} & \\ f_+ \swarrow & & \searrow f_- \\ X_+ & \overset{\varphi}{\dashrightarrow} & X_- \\ g_+ \searrow & & \swarrow g_- \\ & \overline{X}_0 & \end{array}$$

This shows that $f_+^*(K_{X_+}) = f_-^*(K_{X_-})$ and birational map φ is *crepant*, since they are the pull-backs of the same \mathbb{Q} -Cartier divisor on \overline{X}_0 .

To construct \tilde{X} , consider the action of $K \times \mathbb{C}^\times$ on \mathbb{C}^{m+1} defined by the characters $\tilde{D}_1, \dots, \tilde{D}_{m+1}$ of $K \times \mathbb{C}^\times$, where:

$$\tilde{D}_j = \begin{cases} D_j \oplus 0 & \text{if } j < m+1 \text{ and } D_j \cdot e \leq 0 \\ D_j \oplus (-D_j \cdot e) & \text{if } j < m+1 \text{ and } D_j \cdot e > 0 \\ 0 \oplus 1 & \text{if } j = m+1 \end{cases}$$

Consider the chambers \tilde{C}_+ , \tilde{C}_- , and \tilde{C} in $(\mathbb{L} \oplus \mathbb{Z})^\vee \otimes \mathbb{R}$ that contain, respectively, the stability conditions

$$\tilde{\omega}_+ = (\omega_+, 1) \quad \tilde{\omega}_- = (\omega_-, 1) \quad \text{and} \quad \tilde{\omega} = (\omega_0, -\varepsilon)$$

where ε is a very small positive real number. Let \tilde{X} denote the toric DM stack defined by the stability condition $\tilde{\omega}$. We have, by [8, Lemma 6.16], that the toric DM stack corresponding to the chamber \tilde{C}_\pm is X_\pm . Furthermore, there is a commutative diagram as in (2.3), where: $f_\pm: \tilde{X} \rightarrow X_\pm$ is a toric blow-up, arising from the wall-crossing from \tilde{C} to \tilde{C}_\pm .

3. GENERALIZED BONDAL-ORLOV TRANSFORMS

3.1. Equivariant K -theory of toric DM stacks. The big torus $T := (\mathbb{C}^\times)^m$ acts on the toric DM stack X_ω corresponding to a stability condition $\omega \in \mathbb{L}^\vee \otimes \mathbb{R}$ satisfying assumptions (A1-A2). The T -equivariant K -theory group $K_0^T(X_\omega)$ of X_ω is generated by the T -equivariant line bundles R_i corresponding to the ray ρ_i for each $i \in \{1, \dots, m\}$.

Recall that the torus fixed points of X_ω are in one-to-one correspondence with minimal anticones $\delta \in \mathcal{A}_\omega$. A minimal anticone δ determines a torus fixed point stack $x_\delta =$

$BG_\delta \in X_\omega$, where G_δ is the isotropy group of the fixed point x_δ . Let $i_\delta: x_\delta \rightarrow X_\omega$ denote the inclusion. We have

$$(3.1) \quad i_\delta^* R_j = 1, \quad \forall j \in \delta.$$

We recall the Lefschetz fixed point theorem (c.f. [9, Theorem 3.3] in this formulation).

Theorem 3.1. *Let $X_\omega = [U_\omega/K]$ be a toric DM stack. The torus T acts on X_ω . Given $\delta \in \mathcal{A}_\omega$, write x_δ for the corresponding T -fixed point of X_ω . Let N_δ denote the normal bundle to i_δ . Let $\mathbb{Z}[T] = K_T^0(pt)$ denote the ring of regular functions (over \mathbb{Z}) on T and let $\text{Frac } \mathbb{Z}[T]$ denote the field of fractions. Then for $\alpha \in K_T^0(X_\omega)$, we have*

$$\alpha = \sum_{\delta \in \mathcal{A}_\omega} (i_\delta)_* \left(\frac{i_\delta^* \alpha}{\lambda_{-1} N_\delta^\vee} \right) \in K_T^0(X_\omega) \otimes_{\mathbb{Z}[T]} \text{Frac}(\mathbb{Z}[T])$$

where $\lambda_{-1} N_\delta^\vee := \sum_{i=0}^{\dim X_\omega} (-1)^i \wedge^i N_\delta^\vee$ is invertible in $K_T^0(x_\delta) \otimes_{\mathbb{Z}[T]} \text{Frac}(\mathbb{Z}[T])$.

3.2. The localized K-theory basis. Consider the toric wall crossing diagram (2.3). The torus T acts on X_\pm through the diagonal action of T on \mathbb{C}^m . There is an action of T on \tilde{X} induced from the inclusion $T = T \times \{1\} \subset T \times \mathbb{C}^\times$ and the $T \times \mathbb{C}^\times$ action on \mathbb{C}^{m+1} . So all the maps in (2.3) are T -equivariant. The T -equivariant K-groups $K_0^T(X_\pm), K_0^T(\tilde{X})$ are modules over $K_0^T(pt) = \mathbb{Z}[T]$.

From the wall crossing construction in §2.2, there are two types of minimal anticones for \tilde{X} . The first type, called *flopping type*, is given by $\tilde{\delta} = (j_1, \dots, j_{r-1}, j_+, j_-)$, where $j_1, \dots, j_{r-1} \in M_0$, and $j_+ \in M_+, j_- \in M_-$. This type of minimal anticones induce the maps from the fixed point stack of \tilde{X} to the fixed point stacks of X_+ and X_- by

$$f_{+, \tilde{\delta}}: x_{\tilde{\delta}} \rightarrow x_{\delta_+}, \quad f_{-, \tilde{\delta}}: x_{\tilde{\delta}} \rightarrow x_{\delta_-},$$

where $\delta_+ = (j_1, \dots, j_{r-1}, j_+, m+1)$ and $\delta_- = (j_1, \dots, j_{r-1}, j_-, m+1)$. We use the following notations: $\tilde{\delta}|\delta_\pm$ means that the fixed point $x_{\tilde{\delta}}$ maps to the fixed point x_{δ_\pm} corresponding to flopping minimal anticone δ_\pm for X_\pm .

The second type of minimal anticone, called *nonflopping type*, is given by $\tilde{\delta}$ containing the last, $m+1$ -st, ray corresponding to the common blow-up. The nonflopping minimal anticones map isomorphically to minimal anticones of X_+ and X_- . Such minimal anticones ($\tilde{\delta}$ and δ_\pm) are of the form $(j_1, \dots, j_{r-2}, j_+, j_-, m+1)$.

The T -invariant divisor $\{z_i = 0\}$ on X_ω determines a T -equivariant line bundle $\mathcal{O}(\{z_i = 0\})$ on X_ω , and we denote the class of this line bundle in the T -equivariant K-theory by R_i . For $K_0^T(X_\pm), K_0^T(\tilde{X})$ we write these classes as:

$$\begin{aligned} \{R_i^- | 1 \leq i \leq m\} &: & \text{for } K_0^T(X_-); \\ \{R_i^+ | 1 \leq i \leq m\} &: & \text{for } K_0^T(X_+); \\ \{\tilde{R}_i | 1 \leq i \leq m+1\} &: & \text{for } K_0^T(\tilde{X}). \end{aligned}$$

From §6.3.2 in [8], each character $p \in \mathbb{H}om(K, \mathbb{C}^\times) = \mathbb{L}^\vee$ define a line bundle $L_-(p)$ over X_- . This line bundle $L_-(p)$ is equipped with a T -linearized action, thus make it a T -equivariant line bundle. The line bundles $R_i^- = L_-(D_i) \otimes e^{\lambda_i}$, where e^{λ_i} is the standard i -th irreducible T -representation $T \rightarrow \mathbb{C}^\times$. Similar construction works for the K-theory ring $K_0^T(X_+)$.

For a character $(p, n) \in \mathbb{H}om(K \times \mathbb{C}^\times, \mathbb{C}^\times) = \mathbb{L}^\vee \oplus \mathbb{Z}$ we define a T -equivariant line bundle $L(p, n) \rightarrow \tilde{X}$ and we have:

$$\tilde{R}_i = L(\tilde{D}_i) \otimes e^{\lambda_i}, (1 \leq i \leq m); \quad \tilde{R}_{m+1} = L(\tilde{D}_{m+1}) = L(0, 1).$$

The classes $L_\pm(X_\pm)$ (the classes $L(p, n)$) generate the equivariant K -group $K_0^T(X_\pm)$ ($K_0^T(\tilde{X})$) over $\mathbb{Z}[T]$.

We describe the localized T -equivariant K -theory basis for $K_0^T(X_-)$. Let $\delta_- \in \mathcal{A}_-$ be a minimal cone and x_{δ_-} be the corresponding T -fixed point. Let

$$i_{\delta_-} : x_{\delta_-} \rightarrow X_-$$

be the inclusion of the fixed point, and G_{δ_-} the isotropy group of x_{δ_-} . We have $x_{\delta_-} = BG_{\delta_-}$. A basis for $K_0^T(X_-)$, after inverting nonzero elements of $\mathbb{Z}[T]$, is given by

$$(3.2) \quad \{(i_{\delta_-})_* \varrho : \varrho \text{ an irreducible representation of } G_{\delta_-}, \delta_- \in \mathcal{A}_-\}$$

Choose a lift $\hat{\varrho} \in \mathbb{H}om(K, \mathbb{C}^\times) = \mathbb{L}^\vee$ of each G_{δ_-} -representation $\varrho : G_{\delta_-} \rightarrow \mathbb{C}^\times$, an element in (3.2) can be written in the form:

$$e_{\delta_-, \varrho} := L_-(\hat{\varrho}) \prod_{i \notin \delta_-} (1 - S_i^-).$$

Then $\{e_{\delta_-, \varrho}\}$ is a basis for the localized T -equivariant K -theory of X_- . There is a similar basis $\{e_{\delta_+, \varrho}\}$ for the localized T -equivariant K -theory of X_+ .

3.3. The Bondal-Orlov functors. The general Bondal-Orlov functor on the bounded derived categories $D_T^b(X_\pm)$:

$$\mathbb{B}O_k : D_T^b(X_-) \rightarrow D_T^b(X_+)$$

is defined by:

$$\mathbb{B}O_k(\alpha) = (f_+)_*(\mathcal{O}_{\tilde{X}}(kE) \otimes (f_-)^*(\alpha)).$$

We consider the induced functor on the K -theory of X_\pm :

$$\mathbb{B}O_k : K_0^T(X_-) \rightarrow K_0^T(X_+).$$

We explicitly calculate $\mathbb{B}O_k$ in terms of the localized T -equivariant K -theory basis for X_- . Let

$$S_i^+ := (R_i^+)^{-1}, \quad S_i^- := (R_i^-)^{-1}, \quad \tilde{S}_i := (\tilde{R}_i)^{-1},$$

and let

$$k_i := \max(D_i \cdot e, 0), \quad l_i := \max(-D_i \cdot e, 0).$$

Theorem 3.2. *Let $\delta_- \in \mathcal{A}_-$ be a minimal anticone such that $\delta_- \in \mathcal{A}_+$, then $\mathbb{B}O_k(e_{\delta_-, \varrho}) = e_{\delta_-, \varrho}$, where on the right side δ_- is taken as a minimal anticone in \mathcal{A}_+ ; If $\delta_- \in \mathcal{A}_-$ is a minimal anticone such that $\delta_- \notin \mathcal{A}_+$, then*

$$\begin{aligned} \mathbb{B}O_k(e_{\delta_-, \varrho}) = \\ \frac{1}{l} \sum_{t \in T} \left(\frac{t^k (1 - S_{j_-}^+)}{1 - t^{-1}} \cdot L_+(\hat{\varrho}) t^{\hat{\varrho} \cdot e} \cdot \prod_{\substack{j \notin \delta_- \\ D_j \cdot e < 0}} (1 - S_j^+) \cdot \prod_{\substack{i \notin \delta_- \\ D_i \cdot e \geq 0}} (1 - t^{-D_i \cdot e} S_i^+) \right) \end{aligned}$$

where $j_- \in \delta_-$ is the unique element such that $D_{j_-} \cdot e < 0$, $l = -D_{j_-} \cdot e$ and

$$\mathcal{T} := \{\zeta \cdot (R_{j_-}^+)^{\frac{1}{l}} : \zeta \in \mu_l\}.$$

Proof. The proof is similar to the proof of Theorem 6.19 in [8], except that we take into account of the role of the line bundle $\mathcal{O}_{\tilde{X}}(kE)$. The line bundle $\mathcal{O}_{\tilde{X}}(E)$ corresponds to the line bundle \tilde{R}_{m+1} over \tilde{X} . So

$$\mathcal{O}_{\tilde{X}}(kE) \cong \tilde{R}_{m+1}^{\otimes k}.$$

We calculate $\mathbb{B}\mathcal{O}_k$ for any $k \in \mathbb{Z}$. For $\delta_- \in \mathcal{A}_{\pm}$, φ is an isomorphism in an neighbourhoods of the fixed points of $x_{\delta_-} \in X_{\pm}$. So $\mathbb{B}\mathcal{O}_k(e_{\delta_-, \varphi}) = e_{\delta_-, \varphi}$.

Suppose now that $\delta_- \in \mathcal{A}_-$, but $\delta_- \notin \mathcal{A}_+$. Let $\delta_- = \{j_1, \dots, j_{r-1}, j_-\}$. Then $D_{j_1} \cdot e = D_{j_2} \cdot e = \dots = D_{j_{r-1}} \cdot e = 0$ and $D_{j_-} \cdot e < 0$. We have from [8, Proposition 6.21],

$$(f_-)^*(e_{\delta_-, \varphi}) = L(\hat{\varrho}, 0) \prod_{i \notin \delta_-} (1 - \tilde{S}_{m+1}^{k_i} \tilde{S}_i).$$

Then

$$\mathcal{O}_{\tilde{X}}(kE) \otimes (f_-)^*(e_{\delta_-, \varphi}) = \tilde{R}_{m+1}^k \cdot L(\hat{\varrho}, 0) \prod_{i \notin \delta_-} (1 - \tilde{S}_{m+1}^{k_i} \tilde{S}_i).$$

We use the localized Theorem 3.1 in the T -equivariant K -theory restricting above to all torus fixed points $x_{\tilde{\delta}} \in f_-^{-1}(x_{\delta_-})$, where $\tilde{\delta} = \delta_- \cup \{j_+\}$ for $D_{j_+} \cdot e > 0$. So (3.3)

$$\mathcal{O}_{\tilde{X}}(kE) \otimes (f_-)^*(e_{\delta_-, \varphi}) = \sum_{\tilde{\delta} \in \tilde{\mathcal{A}}} (i_{\tilde{\delta}})_* (i_{\tilde{\delta}})^* \left[\frac{\tilde{R}_{m+1}^k \cdot L(\hat{\varrho}, 0) \cdot \prod_{i \notin \delta_-} (1 - \tilde{R}_{m+1}^{k_i} \tilde{S}_i)}{(1 - \tilde{S}_{m+1}) \prod_{j \notin \delta_-} (1 - \tilde{S}_j)} \right]$$

For $j_+ \in \tilde{\delta}$, \tilde{R}_{j_+} is trivial when restricted to $x_{\tilde{\delta}}$. So: $(1 - \tilde{\delta}_{m+1}^{k_i} \tilde{S}_{j_+}) = (1 - \tilde{\delta}_{m+1}^{k_i})$ and (3.3) is actually a polynomial on \tilde{R}_{m+1} on the numerator. Then applying the pushforward

$$\begin{aligned} & (f_+)_* (\mathcal{O}_{\tilde{X}}(kE) \otimes (f_-)^*(e_{\delta_-, \varphi})) \\ &= \sum_{\delta_+ : \delta_+ | \delta_-} (i_{\delta_+})_* (f_{+, \tilde{\delta}})^* (i_{\tilde{\delta}})^* \left[\frac{\tilde{R}_{m+1}^k \cdot L(\hat{\varrho}, 0) \cdot \prod_{i \notin \delta_-} (1 - \tilde{R}_{m+1}^{k_i} \tilde{S}_i)}{(1 - \tilde{S}_{m+1}) \prod_{j \notin \delta_-} (1 - \tilde{S}_j)} \right] \\ &= \sum_{\delta_+ : \delta_+ | \delta_-} (i_{\delta_+})_* (i_{\delta_+})^* \left[\frac{1}{l} \sum_{t \in \mathcal{T}} \frac{t^k \cdot L_+(\hat{\varrho}) \cdot t^{\hat{\varrho} \cdot e} \prod_{i \notin \delta_-} (1 - t^{l_i - k_i} S_i^+)}{(1 - t^{-1}) \prod_{j \notin \delta_-} (1 - t^{l_j} S_j^+)} \right] \end{aligned}$$

here we use the formula (3) in Proposition 6.22 of [8]. Hence we get:

$$\begin{aligned} & (f_+)_* (\mathcal{O}_{\tilde{X}}(kE) \otimes (f_-)^*(e_{\delta_-, \varphi})) \\ &= \sum_{\delta_+ : \delta_+ | \delta_-} (i_{\delta_+})_* (i_{\delta_+})^* \left[\frac{\frac{1}{l} \sum_{t \in \mathcal{T}} \frac{t^k \cdot (1 - S_{j_-}^+)}{1 - t^{-1}} \cdot L_+(\hat{\varrho}) \cdot t^{\hat{\varrho} \cdot e} \prod_{i \notin \delta_-} (1 - t^{-k_i} S_i^+)}{\prod_{j \notin \delta_+} (1 - S_j^+)} \right] \end{aligned}$$

By localization again we get the result in the Theorem. The only thing we need to check is that $t^k \cdot (1 - S_{j_-}^+) \cdot L_+(\hat{\varrho}) \cdot t^{\hat{\varrho} \cdot e} \prod_{i \notin \delta_-} (1 - t^{-k_i} S_i^+)$ vanishes on x_{δ} for $\delta \in \mathcal{A}_+ \cap \mathcal{A}_-$. But this is a similar check as in the proof of Theorem 6.19 of [8]. \square

Remark 3.3. In Theorem 6.23 of [8], we prove that $\mathbb{B}\mathcal{O}_0$ actually matches the analytic continuation of I -functions of X_{\pm} . Since the I -functions of X_{\pm} determine the bid quantum cohomology for X_{\pm} , the result of Theorem 6.23 in [8] tells us that the Fourier–Mukai transform preserves the big quantum cohomology of a single toric wall crossing. It is of course interesting to see if the general Bondal-Orlov transforms $\mathbb{B}\mathcal{O}_k$ preserves some analytic continuation of hypergeometric function of X_{\pm} .

4. DERIVED EQUIVALENCE AND SPHERICAL TWISTS

In this section we recall some facts that the general Bondal-Orlov functors give equivalences on the bounded derived categories.

4.1. Derived equivalence. Let $Q := T/K$ be the quotient torus since $K \subset T$ is a subtorus. Both X_+ and X_- carry effective actions of Q . In this section we prove the following:

Theorem 4.1. *Let (2.3) be a toric crepant transformation. Then*

$$\mathbb{B}\mathcal{O}_k : D_Q^b(X_-) \rightarrow D_Q^b(X_+)$$

gives an equivalence on the equivariant bounded derived categories.

Remark 4.2. We use the same proof as in [9, §5], which uses the idea of Halpern-Leistner [17] and Halpern-Leistner-Shipman [18].

Proof. We mainly follow the construction and notations in §5 of [9]. First we recall the variation of the GIT quotients of X_{\pm} and \tilde{X} . They correspond to chambers $\tilde{C}_{\pm}, \tilde{C}$ inside $(\mathbb{L}^{\vee} \oplus \mathbb{Z}) \otimes \mathbb{R}$. We denote by the walls by $W_{+|-}, W_{+|\sim}, W_{-|\sim}$ respectively. Let

$$W_0 = W_{+|-} \cap W_{+|\sim} \cap W_{-|\sim}.$$

There are 7 stability conditions on $W_0, \tilde{C}_{\pm}, \tilde{C}, W_{+|-}, W_{+|\sim}, W_{-|\sim}$ respectively. If we let $V_0 \subset \mathbb{C}^{m+1}$ be the semi-stable locus of W_0 , then

$$V_0 = U_0 \times \mathbb{C} = \mathbb{C}^{m+1} \setminus \left(\bigcup_{I \notin \mathcal{A}_0} \mathbb{C}^I \times \mathbb{C} \right)$$

where U_0 is in §2.2. As in [9], the other 6 stability conditions are as follows:

Location of stability condition	Semi-stable locus
\tilde{C}_+	$V_+ = V_0 \setminus ((\mathbb{C}^{M_{\leq 0}} \times \mathbb{C}) \cup \mathbb{C}^m)$
\tilde{C}_-	$V_- = V_0 \setminus ((\mathbb{C}^{M_{\geq 0}} \times \mathbb{C}) \cup \mathbb{C}^m)$
\tilde{C}	$V_{\sim} = V_0 \setminus ((\mathbb{C}^{M_{\leq 0}} \times \mathbb{C}) \cup (\mathbb{C}^{M_{\geq 0}} \times \mathbb{C}))$
$W_{+ -}$	$V_{+ -} = V_0 \setminus \mathbb{C}^m$
$W_{+ \sim}$	$V_{+ \sim} = V_0 \setminus (\mathbb{C}^{M_{\leq 0}} \times \mathbb{C})$
$W_{- \sim}$	$V_{- \sim} = V_0 \setminus (\mathbb{C}^{M_{\geq 0}} \times \mathbb{C})$

We have the GIT quotients

$$\begin{aligned} X_+ &= [V_+/K], \\ X_- &= [V_-/K], \\ \tilde{X} &= [V_{\sim}/K]. \end{aligned}$$

Now we recall the *KN* stratum introduced in [9, §5.1]. A *KN* stratum (λ, Z, S) contains a one-parameter subgroup $\lambda \subset K \times \mathbb{C}^\times$, a connected component Z of the fixed locus, and the associated blade S defined as:

$$S = \{y \in \mathbb{C}^{m+1} : \lim_{t \rightarrow \infty} \lambda(t)(y) \in Z\}.$$

To a *KN* stratum, there is a numerical invariant

$$\eta = \text{Weight}_\lambda(\det(N_{S/\mathbb{C}^{m+1}})).$$

In our cases let

$$d := \sum_{i \in M_+} D_i \cdot e = - \sum_{i \in M_-} D_i \cdot e$$

and consider the *KN*-strata:

$$((e, 1), \mathbb{C}^{M_{\geq 0}} \cap V_{+|\sim}, \mathbb{C}^m \cap V_{+|\sim}), \quad \eta = 1$$

and

$$((-e, -1), \mathbb{C}^{M_{\geq 0}} \cap V_{+|\sim}, \mathbb{C}^m \cap V_{+|\sim}), \quad \eta = d$$

Then V_+ and V_- are open subsets of $V_{+|\sim}$, which are the complements of the above *KN* strata. Then from [9, §5] and [17], let

$$\mathbf{F} \subset \tilde{\mathbf{F}} \subset D_{T \times \mathbb{C}^\times}^b(V_{+|\sim})$$

be the subcategories by imposing the grade-restriction rule on the subvariety $\mathbb{C}^{M_{\geq 0}} \cap V_{+|\sim}$, where for \mathbf{F} we require that the $(e, 1)$ -weights lie in $[0, 1)$, and for $\tilde{\mathbf{F}}$ we require that the $(e, 1)$ -weights lie in $[0, d)$. Then we have the following diagram:

$$\begin{array}{ccc} & \tilde{\mathbf{F}} & \\ \cong \swarrow & & \searrow \\ D_Q^b(\tilde{X}) & \xrightarrow{(f_+)^*} & D_Q^b(X_+) \end{array}$$

and the diagonal map is the restriction of functors. Similarly, take V_- as an open subset of $V_{-|\sim}$ and taking into account of the *KN*-stratum:

$$((0, 1), \mathbb{C}^{M_{\leq 0}} \cap V_{-|\sim}, \mathbb{C}^m \cap V_{-|\sim})$$

which has numerical invariant $\eta = 1$. There is a subcategory

$$\mathbf{H} \subset D_{T \times \mathbb{C}^\times}^b(V_{-|\sim})$$

such that the $(0, 1)$ -weights lie in $[0, 1)$. We have the commuting triangle:

$$\begin{array}{ccc} & \mathbf{H} & \\ \cong \swarrow & & \searrow \\ D_Q^b(X_-) & \xrightarrow{(f_-)^*} & D_Q^b(\tilde{X}) \end{array}$$

and the diagonal map is the restriction of functors

Let us recall the definition of the functor GR_k for each integer $k \in \mathbb{Z}$ in [9, §5.1]. Note that [9] only discusses the case GR_0 , but general GR_k are similar. For the *KN* stratum (e, Z, S_-) with numerical invariant $\eta_+ = \sum_{i \in M_+} D_i \cdot e$, $Z = U_0 \cap \mathbb{C}^{M_0}$ and

$S_- = U_0 \cap \mathbb{C}^{M_{\leq 0}}$, where the toric DM stack $X_+ = [(U_0 \setminus S_-)/K]$, there exists a subcategory

$$\mathbf{G}_k \subset D_T^b(U_0)$$

using the grade restriction rule and requiring the e -weights lying in $[k, k + \eta_+)$. We have $\mathbf{G}_k \cong D^b(X_+)$.

On the other hand, for the KN stratum $(-e, Z, S_+)$ with numerical invariant $\eta_- = -\sum_{i \in M_-} D_i \cdot e$, $Z = U_0 \cap \mathbb{C}^{M_0}$ and $S_+ = U_0 \cap \mathbb{C}^{M_{\geq 0}}$, where the toric DM stack $X_- = [(U_0 \setminus S_+)/K]$, there exists a subcategory

$$\mathbf{G}_k \subset D_T^b(U_0)$$

using the grade restriction rule and requiring the $(-e)$ -weights lying in $[-\eta_- + k + 1, k + 1)$. Then we have $\mathbf{G}_k \cong D^b(X_-)$. Thus the functor $\mathrm{GR}_k : D_Q^b(X_-) \rightarrow D_Q^b(X_+)$ are defined by the diagram:

$$\begin{array}{ccc} & \mathbf{G}_k & \\ \cong \swarrow & & \searrow \cong \\ D_Q^b(X_-) & \xrightarrow{\quad} & D_Q^b(X_+) \end{array}$$

by inverting the right isomorphism.

Consider the subcategory $(\pi_-)^* \mathbf{G}_0 \subset D_{T \times \mathbb{C}^\times}^b(V_0)$, where

$$\pi_- : [V_0 / (T \times \mathbb{C}^\times)] \rightarrow [U_0 / T]$$

is the natural morphism. Under the restriction functor from $V_0 \rightarrow V_{+|\sim}$, the subcategory $(\pi_-)^* \mathbf{G}_0$ maps to $\tilde{\mathbf{F}}$. Under the restriction functor from $V_0 \rightarrow V_{-|\sim}$, the subcategory $(\pi_-)^* \mathbf{G}_0$ maps to \mathbf{H} , which is an isomorphism.

The line bundle $\mathcal{O}_{\tilde{X}}(kE) \rightarrow \tilde{X}$ corresponds to an $T \times \mathbb{C}^\times$ -equivariant line bundle \mathcal{L}_k on \mathbb{C}^{m+1} . Let

$$\otimes \mathcal{L}_k : D_Q^b(\tilde{X}) \rightarrow D_Q^b(\tilde{X})$$

be the tensor product morphism. Then since the line bundle has e -weight k , the tensor product sends $(\pi_-)^* \mathbf{G}_0$ to $(\pi_-)^* \mathbf{G}_k$. We have the following modified diagram as for the last diagram in [9]:

$$\begin{array}{ccccccc} & & (\pi_-)^* \mathbf{G}_0 & \xrightarrow{\otimes \mathcal{L}_k} & (\pi_-)^* \mathbf{G}_k & & \\ & \swarrow \cong & & & \searrow & & \\ & \mathbf{H} & & & \tilde{\mathbf{F}} & & \\ \swarrow \cong & & & & \swarrow \cong & & \\ D_Q^b(X_-) & \xrightarrow{(f_-)^*} & D_Q^b(\tilde{X}) & \xrightarrow{\otimes \mathcal{L}_k} & D_Q^b(\tilde{X}) & \xrightarrow{(f_+)^*} & D_Q^b(X_+) \end{array}$$

The result is easily seen from the above diagram since the bottom represents the Bondal-Orlov functor $\mathbb{B}O_k$. \square

4.2. The spherical twist. Let us fix a single toric wall crossing (2.3). We first classify the exceptional locus of the contractions g_{\pm} .

For $g_+ : X_+ \rightarrow \overline{X}_0$, let

$$\mathbb{L}_{\text{ex}}^{\vee} := \mathbb{L}^{\vee} / \langle D_i : i \in M_0 \rangle$$

and let $p : \mathbb{L}^{\vee} \rightarrow \mathbb{L}_{\text{ex}}^{\vee}$ be the projection. Then $p : \mathbb{L}^{\vee} \otimes \mathbb{R} \rightarrow \mathbb{L}_{\text{ex}}^{\vee} \otimes \mathbb{R}$ is the projection to the vector spaces. Let $\omega_{\text{ex}}^+ = p(\omega_+)$ be the image of the stability condition ω_+ . The lattice $\mathbb{L}_{\text{ex}}^{\vee}$, which is rank one, may have torsion in general. In this section we assume that $\langle D_i : i \in M_0 \rangle$ generate the lattice wall $W \cap \mathbb{L}^{\vee}$. Then $\mathbb{L}_{\text{ex}}^{\vee} \cong \mathbb{Z}$. The elements $D_i \in \mathbb{L}^{\vee}$ have images $p(D_i) = D_i \cdot e \in \mathbb{L}_{\text{ex}}^{\vee}$. So only D_i for $i \in M_{\pm}$ survive. Hence we get the GIT data on $\mathbb{L}_{\text{ex}}^{\vee}$:

- $K \cong \mathbb{C}^{\times}$, a connected torus of rank 1;
- $\mathbb{L}_{\text{ex}} = \text{Hom}(\mathbb{C}^{\times}, K)$;
- $D_1 \cdot e, \dots, D_m \cdot e \in \mathbb{L}_{\text{ex}}^{\vee} = \text{Hom}(K, \mathbb{C}^{\times})$, characters of K ;
- stability condition $\omega_{\text{ex}}^+ \in \mathbb{L}_{\text{ex}}^{\vee} \otimes \mathbb{R}$;
- $\mathcal{A}_{\omega_{\text{ex}}^+} = \{I \subset \{1, 2, \dots, m\} : D_i \cdot e > 0, i \in I\}$.

Let $a_i := D_i \cdot e$ for $D_i \cdot e > 0$ and $\mathbf{a} = (D_i \cdot e : D_i \cdot e > 0)$.

Proposition 4.3. *The corresponding toric DM stack $X_{\omega_{\text{ex}}^+}$ associated with the above GIT data is the weighted projective stack $\mathbb{P}(\mathbf{a})$. Moreover, the map $g_+ : X_+ \rightarrow \overline{X}_0$ always contracts the weighted projective stack $X_{\omega_{\text{ex}}^+} = \mathbb{P}(\mathbf{a})$.*

Proof. The first statement is easily seen from the GIT data. For the second statement, look at the map

$$g_+ : X_+ = [U_+/K] \rightarrow \overline{X}_0 = \overline{[U_0/K]},$$

where $U_+ = U_0 \setminus (\mathbb{C}^{M_{\leq 0}} \cap U_0)$. The torus \mathbb{C}^{\times} -fixed points on $\overline{X}_0 = \overline{[U_0/K]}$ corresponds to nonsimplicial cones, which are spanned by rays containing D_i 's for $i \in M_{\pm}$. Then from the above map g_+ , it must contract the weighted projective stack $\mathbb{P}(\mathbf{a})$ to this fixed point. \square

Remark 4.4. Similar result holds for the contract map

$$g_- : X_- = [U_-/K] \rightarrow \overline{X}_0 = \overline{[U_0/K]}.$$

Let $b_i := D_i \cdot e$ for $D_i \cdot e < 0$ and $\mathbf{b} = (D_i \cdot e : D_i \cdot e < 0)$. Then g_- contracts the weighted projective stacks $\mathbb{P}(\mathbf{b})$.

Let $N := \sum_{i: D_i \cdot e > 0} D_i \cdot e = -\sum_{i: D_i \cdot e < 0} D_i \cdot e$, which is the sum of the weights.

Proposition 4.5. *We have for any $k \in \mathbb{Z}$,*

$$\mathbb{G}\mathbb{R}_k \cong \mathbb{B}\mathbb{O}_{(N-1)+k}.$$

Proof. We generalize the proof of Proposition 3.1 in [1]. We show that $\mathbb{G}\mathbb{R}_k^{-1} \circ \mathbb{B}\mathbb{O}_{(N-1)+k}$ takes $\mathcal{O}_{X_-}(l)$ to $\mathcal{O}_{X_-}(l)$ and acts as identity on

$$\mathcal{E}xt^i(\mathcal{O}_{X_-}(l), \mathcal{O}_{X_-}(l'))$$

for $k \leq l, l' \leq k + (N-1)$, since these objects split-generate the derived category $D^b(X_-)$. First $\mathbb{G}\mathbb{R}_k$ takes $\mathcal{O}_{X_-}(l)$ to $\mathcal{O}_{X_-}(-l)$ for $k \leq l \leq k + (N-1)$, since the subcategory $\mathbf{G}_k \subset D_T^b(U_0) = D^b(X_0)$ is the full-subcategory split-generated by

$$\mathcal{O}_{X_0}(k), \dots, \mathcal{O}_{X_0}(k + (N-1)).$$

Also

$$\begin{aligned} \mathbb{B}\mathcal{O}_{(N-1)+k}(\mathcal{O}_{X_-}(l)) &= (f_+)_* (\mathcal{O}_{\tilde{X}}(((N-1)+k)E) \otimes (f_-)^*(\mathcal{O}_{X_-}(l))) \\ &= (f_+)_* (\mathcal{O}_{\tilde{X}}(l - (N-1) - k, -(N-1) - k)) \\ &= \mathcal{O}_{X_+}(-l) \otimes (f_+)_* (\mathcal{O}_{\tilde{X}}(((N-1)+k-l)E)) \end{aligned}$$

and

$$(f_+)_* (\mathcal{O}_{\tilde{X}}(((N-1)+k-l)E)) \cong \mathcal{O}_{X_+}$$

for $0 \leq (N-1)+k \leq (N-1)$. So $\mathbb{G}\mathbb{R}_k^{-1} \circ \mathbb{B}\mathcal{O}_{(N-1)+k}$ takes $\mathcal{O}_{X_-}(l)$ to $\mathcal{O}_{X_-}(l)$ for $k \leq l \leq k + (N-1)$. The proof that $\mathbb{G}\mathbb{R}_k^{-1} \circ \mathbb{B}\mathcal{O}_{(N-1)+k}$ acts as identity on $\mathcal{E}xt^i(\mathcal{O}_{X_-}(l), \mathcal{O}_{X_-}(l'))$ is the same as [1, Proposition 3.1]. \square

Let

$$j_+ : \mathbb{P}(\mathbf{a}) \hookrightarrow X_+; \quad j_- : \mathbb{P}(\mathbf{b}) \hookrightarrow X_-$$

be the closed immersions for the weighted projective stacks $\mathbb{P}(\mathbf{a})$ and $\mathbb{P}(\mathbf{b})$. To abuse notations, we understand $\mathcal{O}_{\mathbb{P}(\mathbf{b})}(k)$ as line bundle over $\mathbb{P}(\mathbf{b})$, and at the same time taken as the coherent sheaf $j_{-*}\mathcal{O}_{\mathbb{P}(\mathbf{b})}(k)$ on X_- . The same situation holds for $j_+ : \mathbb{P}(\mathbf{a}) \hookrightarrow X_+$.

Proposition 4.6. *We have the following result for the autoequivalence:*

$$\mathbb{G}\mathbb{R}_k^{-1} \circ \mathbb{G}\mathbb{R}_{k+1} = \mathbb{T}_{\mathcal{O}_{\mathbb{P}(\mathbf{b})}(k)}$$

associated with the spherical functor

$$\mathbb{T}_{\mathcal{O}_{\mathbb{P}(\mathbf{b})}(k)} : D_Q^b(X_-) \rightarrow D_Q^b(X_-)$$

defined by:

$$\mathbb{T}_{\mathcal{O}_{\mathbb{P}(\mathbf{b})}(k)}(\mathcal{E}) = \text{Cone}(\mathcal{O}_{\mathbb{P}(\mathbf{b})}(k) \otimes \text{RHom}(\mathcal{O}_{\mathbb{P}(\mathbf{b})}(k), \mathcal{E}) \xrightarrow{\text{eval}} \mathcal{E}).$$

Proof. We generalize the proof in Proposition 3.2 of [1]. We prove the $k = 0$ case, since other cases are similar. It suffices to check that both functors act on $\mathcal{O}_{X_-}(1), \dots, \mathcal{O}_{X_-}(N)$, since these objects split-generate the derived category $D_Q^b(X_-)$. Clearly $\mathbb{G}\mathbb{R}_0^{-1} \circ \mathbb{G}\mathbb{R}_1$ and $\mathbb{T}_{\mathcal{O}_{\mathbb{P}(\mathbf{b})}}$ act on $\mathcal{O}_{X_-}(1), \dots, \mathcal{O}_{X_-}(N-1)$ as identities. This is due to the facts that the full subcategories $\mathbf{G}_0 \subset D_Q^b(X_0)$; and $\mathbf{G}_1 \subset D_Q^b(X_0)$ are split-generated by the objects $\mathcal{O}_{X_0}, \dots, \mathcal{O}_{X_0}(N-1)$; and $\mathcal{O}_{X_0}(1), \dots, \mathcal{O}_{X_0}(N)$, respectively.

We check the case $\mathcal{O}_{X_-}(N)$. Consider the Koszul resolution of the substack $[\mathbb{C}^{M < 0} \cap U_0/K] \subset X_0$, which is cut out by a transverse section of $\mathcal{O}_{X_0}(-1) \otimes S_+$:

$$\mathcal{O}_{X_0}(N) \otimes \det(S_+^*) \rightarrow \dots \rightarrow \mathcal{O}_{X_0}(2) \otimes \wedge^2(S_+^*) \rightarrow \mathcal{O}_{X_0}(1) \otimes S_+^* \rightarrow \mathcal{O}_{X_0} \rightarrow \mathcal{O}_{[\mathbb{C}^{M < 0} \cap U_0/K]}.$$

Restrict to X_- we get:

$$(4.1) \quad \mathcal{O}_{X_-}(N) \otimes \det(S_+^*) \rightarrow \dots \rightarrow \mathcal{O}_{X_-}(2) \otimes \wedge^2(S_+^*) \rightarrow \mathcal{O}_{X_-}(1) \otimes S_+^* \rightarrow \mathcal{O}_{X_-} \rightarrow \mathcal{O}_{j_{-*}\mathcal{O}_{\mathbb{P}(\mathbf{b})}}.$$

Then we restrict to X_+ , we get: (Note that there is no last term.)

$$(4.2) \quad \mathcal{O}_{X_+}(-N) \otimes \det(S_+^*) \rightarrow \dots \rightarrow \mathcal{O}_{X_+}(-2) \otimes \wedge^2(S_+^*) \rightarrow \mathcal{O}_{X_+}(-1) \otimes S_+^* \rightarrow \mathcal{O}_{X_+}.$$

Now we have

$$\mathbb{G}\mathbb{R}_1(\mathcal{O}_{X_-}(N)) = \mathcal{O}_{X_+}(-N).$$

Use (4.2) we get:

$$\underbrace{\mathcal{O}_{X_+}(-(N-1)) \otimes S_+}_{\deg 0} \rightarrow \mathcal{O}_{X_+}(-(N-2)) \otimes \wedge^2(S_+) \rightarrow \cdots \rightarrow \mathcal{O}_{X_+} \otimes \det(S_+)$$

Then applying the functor GR_0 ,

$$\underbrace{\mathcal{O}_{X_-}(N-1) \otimes S_+}_{\deg 0} \rightarrow \mathcal{O}_{X_-}(N-2) \otimes \wedge^2(S_+) \rightarrow \cdots \rightarrow \mathcal{O}_{X_-} \otimes \det(S_+)$$

which is the middle N -terms of (4.1) tensored with $\det(S_+)$, and this extension is

$$\mathrm{Cone}(j_{-\star} \mathcal{O}_{\mathbb{P}(\mathbf{b})} \otimes \det(S_+)[-N] \rightarrow \mathcal{O}_{X_-}(N)).$$

On the other hand, the spherical twist

$$\mathbb{T}_{\mathcal{O}_{\mathbb{P}(\mathbf{b})}}(\mathcal{O}_{X_-}(N)) = \mathrm{Cone}(j_{-\star} \mathcal{O}_{\mathbb{P}(\mathbf{b})} \otimes \mathrm{RHom}(j_{-\star} \mathcal{O}_{\mathbb{P}(\mathbf{b})}, \mathcal{O}_{X_-}(N)) \rightarrow \mathcal{O}_{X_-}(N))$$

has the same description. These two extensions are the same since the functors $\mathrm{GR}_0^{-1} \circ \mathrm{GR}_1$ and $\mathbb{T}_{\mathcal{O}_{\mathbb{P}(\mathbf{b})}}$ acts in the same way on the Exts. \square

Let

$$\mathbb{B}\mathcal{O}'_k : D_Q^b(X_+) \rightarrow D_Q^b(X_-)$$

be the general Bondal-Orlov functors other way around by:

$$\mathbb{B}\mathcal{O}'_k := (f_-)_*(\mathcal{O}_{\tilde{X}}(kE) \otimes (f_+)^*(-)).$$

The degree zero $\mathbb{B}\mathcal{O}'_0$ is the Fourier-Mukai transform $\mathbb{F}\mathcal{M}'$.

Corollary 4.7. *We have:*

$$\mathbb{F}\mathcal{M}' \circ \mathbb{F}\mathcal{M} = \mathbb{T}_{\mathcal{O}_{\mathbb{P}(\mathbf{b})}(-1)}^{-1} \circ \cdots \circ \mathbb{T}_{\mathcal{O}_{\mathbb{P}(\mathbf{b})}(-(N-1))}^{-1}.$$

Proof. The results in Proposition 4.5 and Proposition 4.6 imply that

$$\mathbb{B}\mathcal{O}'_{-k} \circ \mathbb{B}\mathcal{O}_{(N-1)+k+1} = \mathbb{T}_{\mathcal{O}_{\mathbb{P}(\mathbf{b})}(k)}.$$

Hence we have:

$$\mathbb{B}\mathcal{O}'_{-k-1} \circ \mathbb{B}\mathcal{O}_{(N-1)+k} = \mathbb{T}_{\mathcal{O}_{\mathbb{P}(\mathbf{b})}(k)}^{-1}.$$

By Grothendieck duality, we have that $\mathbb{B}\mathcal{O}_k^{-1} = \mathbb{B}\mathcal{O}'_{(N-1)-k}$. Then the result is a direct calculation. \square

Remark 4.8. By a similar argument we have:

$$\mathbb{F}\mathcal{M} \circ \mathbb{F}\mathcal{M}' = \mathbb{T}_{\mathcal{O}_{\mathbb{P}(\mathbf{a})}(-1)}^{-1} \circ \cdots \circ \mathbb{T}_{\mathcal{O}_{\mathbb{P}(\mathbf{a})}(-(N-1))}^{-1}.$$

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